CONSTRUCTING THE STABILITY DOMAIN IN THE PARAMETER SPACE OF A DYNAMIC SYSTEM

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The necessity for constructing in the space of parameters of a system the domain corresponding to the disposition of the roots of the characteristic polynomial inside a unit circle arises in many problems, including those relating to the stability of periodic motions of strongly nonlinear dynamic systems. The relationships between parameters which include the boundaries of the domain of stable solutions can usually be found from the conditions of nonfulfillment of a system of inequalities constructed in a certain way from the coefficients of the characteristic equation. An equation of degree n usually requires the investigation of up to $2n$ "candidates" for the aforementioned boundaries [1 to 5].

In [6 and 7] the author proposes a method requiring the construction of just three "candidates" (N_+, N_-, and N_ϕ) for the boundaries of the stability domain; the equations of these "candidates" are obtained from the characteristic polynomial $\chi(z) = 0$ by substituting into it the values $z = +1$, $z = -1$, and $z = e^{i\phi}$, respectively. In constructing N_ϕ it is necessary to isolate the real and imaginary parts of Expression $\chi(e^{i\phi}) = 0$ and to investigate the corresponding stability boundary in parameteric form ($0 \leqslant \phi \leqslant \pi$). This can be difficult in the case $n \geqslant 3$ (see [8 to 11]). The relationship between the coefficients of $\chi(z)$ which include the boundary N_ϕ constructed in [12] does not retain the advantages of the parametric form of definition, i.e. the conditions of isolation of the parasitic part and the shading rule.

We propose to derive the equation for N_ϕ in the form of an explicit relationship among the coefficients of the characteristic polynomial which retains the above advantages. We shall also consider the structure of the parameter space in the neighborhood of certain special configurations whose equations are more readily amenable to investigation than the equations of N-surfaces. This enables us to simplify the construction of the stability domain and to investigate (in certain cases) the dependence of stability on the parameters.

1. Let the characteristic equation

$$\chi(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0 \tag{1.1}$$

for certain values of its real coefficients have two conjugate roots lying on a circle of unit radius. In this case Eq. (1.1) must have as one of its factors the product

$$(z - e^{i\varphi})(z - e^{-i\varphi}) = z^2 + pz + 1$$

in which the real parameter $p \in (-2, +2)$. Hence, the values of the coefficients a_k satisfy the equation of N_ϕ if Eq.

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = (z^2 + pz + 1)(b_2 z^{n-2} + \cdots + b_{n-1} z + b_n) \tag{1.2}$$

is fulfilled.

Equating terms with equal powers of z in (1.2) and successively eliminating the coefficients $b_n, \ldots, b_2$ from the system, we arrive at the following parametric Eqs. for N_ϕ:

$$F_0(p) = a_1 + a_2 f_1(p) + a_3 f_2(p) + \ldots + a_n f_{n-1}(p) = 0$$
$$\Phi_0(p) = a_0 + a_1 f_1(p) + a_2 f_2(p) + \ldots + a_n f_n(p) = 0 \quad (1.3)$$

By $f_k(p)$ ($k = 1, 2, \ldots, n$) we denote polynomials of degree k in p,

$$f_1 = -p, \quad f_2 = -1 - pf_1, \quad f_3 = -f_1 - pf_2, \ldots, f_n = -f_{n-2} - pf_{n-1} \quad (1.4)$$

In the case of two roots, $e^{i\phi}$ and $e^{-i\phi}$, we require fulfillment not only of (1.3), but also of the following Eqs. which result from (1.3) upon the substitution of coefficients $a_0 \to a_n$, $\ldots, a_k \to a_{n-k}, \ldots, a_n \to a_0$:

$$F_0^*(p) = a_{n-1} + a_{n-2} f_1(p) + \cdots + a_0 f_{n-1}(p) = 0$$
$$\Phi_0^*(p) = a_n + a_{n-1} f_1(p) + \cdots + a_0 f_n(p) = 0 \quad (1.5)$$

In order to obtain the equation for N_ϕ in the form of an explicit relationship among the coefficients a_k, we eliminate p by successively reducing Eqs. (1.3) and (1.5) with the aid of the transformations

$$\Phi_{j+1} = a_0^{(j)} \Phi_j - a_{n-j}^{(j)} \Phi_j^*, \qquad F_{j+1} = a_0^{(j)} F_j - a_{n-j}^{(i)} F_j^*$$
$$\Phi_{j+1}^* = a_0^{(j)} F_j^* - a_{n-j}^{(j)} F_j, \qquad F_{j+1}^* = a_0^{(j)} \Phi_j^* - a_{n-j}^{(j)} \Phi_j \qquad (a_k^{(0)} = a_k) \quad (1.6)$$

After each transformation (1.6), systems (1.3) and (1.5) contain polynomials $f_k(p)$ lower by one degree. The coefficients $a_k^{(j)}$ here coincide exactly with the corresponding coefficients obtained in constructing the Schur inequalities [13 and 14]. After $n - 2$ reductions (1.6) we arrive at Eqs.

$$a_1^{(n-2)} - pa_2^{(n-2)} = 0, \qquad a_0^{(n-2)} - pa_1^{(n-2)} + (p^2 - 1)\, a_2^{(n-2)} = 0$$
$$a_1^{(n-2)} - pa_0^{(n-2)} = 0, \qquad a_2^{(n-2)} - pa_1^{(n-2)} + (p^2 - 1)\, a_0^{(n-2)} = 0 \quad (1.7)$$

Eliminating p from (1.7), we obtain Eq. of N_ϕ,

$$a_2^{(n-2)} - a_0^{(n-2)} = 0, \qquad |a_1^{(n-2)} / a_0^{(n-2)}| \leqslant 2 \quad (1.8)$$

The additional inequality $|p| \leqslant 2$ isolates on the surface $a_2^{(n-2)} = a_0^{(n-2)}$ the portion which is the boundary N_ϕ from the so-called parasitic part corresponding to the roots $z_1 z_2 = 1$ which do not lie on the unit circle.

According to Schur's rule, the necessary and sufficient conditions whereby all the roots of polynomial (1.1) lie inside the unit circle are the inequalities

$$\left|\frac{a_n}{a_0}\right| < 1, \quad \left|\frac{a_{n-1}^{(1)}}{a_0^{(1)}}\right| < 1, \ldots, \left|\frac{a_2^{(n-2)}}{a_0^{(n-2)}}\right| < 1, \quad \left|\frac{a_1^{(n-1)}}{a_0^{(n-1)}}\right| < 1$$

Hence, in constructing the boundaries of the stability domain it is sufficient to consider the violation of the penultimate condition of Schur's inequality, when $a_2^{n-2} / a_0^{n-2} = +1$ (this is associated with the appearance of the pair of complex conjugate roots $e^{i\phi}$, $e^{-i\phi}$) and the two relations $\chi(+1) = 0$ and $\chi(-1) = 0$ associated with the appearance of the roots $z = +1$ and $z = -1$.

Clearly, the stability domain can lie only on that side of N_ϕ for which the penultimate Schur inequality, i.e. $a_2^{n-2}/a_0^{n-2} < 1$, is fulfilled strictly.

The above inequality defines the shading rule.

In constructing the characteristic equation in the form of a determinant we can obtain the penultimate Schur inequality without reducing the equation to polynomial form using the formulas of [14].

For example, let us write out the equations of the surfaces N_+, N_-, and N_ϕ for a characteristic third-degree polynomial (1.1),

$$a_0 + a_1 + a_2 + a_3 = 0, \; -a_0 + a_1 - a_2 + a_3 = 0$$
$$a_3(a_3 - a_1) - a_0(a_0 - a_2) = 0, \quad |(a_3 - a_0) / a_3| < 2 \quad (1.9)$$

The side of N_ϕ corresponding to the entry of the roots $e^{\pm i\phi}$ into the unit circle is defined by the inequality

$$\frac{a_0a_2 - a_1a_3}{a_0^2 - a_3^2} < 1 \tag{1.10}$$

Eqs. of N_+, N_-, and N_ϕ for a fourth-degree polynomial are:

$$a_0 + a_1 + a_2 + a_3 + a_4 = 0, \qquad a_0 - a_1 + a_2 - a_3 + a_4 = 0$$

$$(a_0 - a_4)^2 (a_0 + a_4 - a_2) - (a_1 - a_3)(a_4a_1 - a_0a_3) = 0, \quad |(a_3 - a_1)/(a_4 - a_0)| \leqslant 2 \tag{1.11}$$

The side of N_ϕ corresponding to the entry of the roots $e^{\pm i\phi}$ into the unit circle is defined by the inequality

$$\frac{a_2(a_0^2 - a_4^2)(a_0 - a_4) - (a_0a_1 - a_3a_4)(a_0a_3 - a_1a_4)}{(a_0^2 - a_4^2)^2 - (a_0a_3 - a_1a_4)^2} < 1 \tag{1.12}$$

2. The hypersurfaces N_+, N_-, and N_ϕ divide the parameter space of the dynamic system into domains D_k, where the subscript k denotes the number of roots of the characteristic polynomial inside the unit circle. The general method of solving the problem of D-decomposition with respect to the unit circle is described in [6 and 7]. In constructing just the stability domain D_n there is no need to construct those portions of the N-surfaces which are not its boundaries. In some cases the determination of the position of the stability domain and the investigation of the qualitative dependence of stability on the parameters are facilitated substantially by a knowledge of the structure of the parameter space in the neighborhood of multiple configurations whose equations are usually more amenable to analysis than the equations of the N-surfaces.

We shall call the points of the parameter space of a dynamic system "parametric points". By an s-tuple point we mean a point lying on the boundary between the domains $D_k, \ldots, D_{k+s}$ the maximum difference between whose subscripts is s. Construction of the stability domain can be conveniently begun with the determination of the multiple points and the investigation of the values at these points of the remaining $n - s$ roots of $\chi(z) = 0$. If these roots lie inside the unit circle or if $s = n$, then the stability domain lies in the neighborhood of an s-tuple point. It must be noted that in considering D_k in the parameter space of a dynamic system the value of s depends substantially on the actual choice of variable parameters and of the way in which the coefficients of the polynomial depend on these parameters. It is clear that by suitable choice of one of the variable parameters (in passing from the consideration of a multidimensional parameter space to the investigation of a one-dimensional space) an s-tuple parametric point can be made a 0-tuple point. For this reason the consideration of multiple s-configurations must be related to a specific dimensionality of the parameter space.

Omitting proofs, let us formulate the characteristics of certain multiple points of the dynamic system parameter space $\mu, \lambda, \nu, \ldots$ We shall assume that the function $\chi(z, \mu, \lambda, \ldots)$ is continuous and that it can be differentiated the required number of times with respect to z and with respect to the parameters $\mu, \lambda, \ldots$

1°. A parametric point of the surface N_+ or N_- is singular in the one-dimensional space μ if at this point

$$\chi_z' \neq 0, \qquad \chi_\mu' \neq 0 \tag{2.1}$$

The change in the parameter μ associated with transition from the domain D_k into the domain D_{k+1} satisfies the condition

$$(\chi_z' \chi_\mu' z)_{z=\pm 1}\, d\mu > 0 \tag{2.2}$$

2°. A parametric point of the surface N_ϕ is double in the one-dimensional space μ if at this point

$$\chi_{|z|}' \neq 0, \qquad \chi_\mu' \neq 0 \tag{2.3}$$

The change in the parameter μ associated with transition from the domain D_k into the domain D_{k+1} satisfies the inequality

$$(\chi_{|z|}{}' \chi_\mu{}')_{|z|=1} d\mu > 0 \tag{2.4}$$

We note that computation of the derivative $\chi_{|z|}'$ can be avoided by determining which side of N_ϕ is associated with D_{k+2} from the shading rule $a_2^{(n-2)}/a_0^{(n-2)} < 1$ or from a consideration of $\chi(z)$ in the neighborhood of a multiple point for a suitably chosen variable parameter μ (see Example 3.2).

3°. A parametric point belonging to the intersection of the surfaces N_+ and N_{-1} is a double point in the plane μ, λ if at this point

$$(\chi_z')_{+1} \neq 0, \qquad (\chi_z')_{-1} \neq 0, \qquad \delta = \begin{vmatrix} (\chi_\mu')_{+1}, & (\chi_\lambda')_{+1} \\ (\chi_\mu')_{-1}, & (\chi_\lambda')_{-1} \end{vmatrix} \neq 0 \tag{2.5}$$

The change in the parameters associated with transition to a domain with a larger number of roots inside the unit circle satisfies the condition

$$\delta (\chi_\lambda')_{+1} (\chi_z')_{-1} d\mu > 0 \qquad \text{for } \lambda = \text{const}, \qquad (\chi_\lambda')_{+1} \neq 0 \tag{2.6}$$

4°. A parametric point of the surface N_+ or N_- is double point in the parameter plane μ, λ if at this point

$$\chi_z' = 0, \qquad \chi_{zz}'' \neq 0, \qquad \Delta = \begin{vmatrix} \chi_\mu', & \chi_\lambda' \\ \chi_{z\mu}'', & \chi_{z\lambda}'' \end{vmatrix} \neq 0 \tag{2.7}$$

The surface N_ϕ "begins" at the indicated points of the surface N_+ or N_-. Transitions from these points into a domain with the largest number of roots inside the unit circle are associated with fulfillment of the condition

$$(z\Delta\chi_\lambda'\chi_{zz}'')_{\pm 1} d\mu < 0 \qquad \text{for } \lambda = \text{const}, \qquad \chi_\lambda' \neq 0 \tag{2.8}$$

5°. A parametric point of intersection of the surfaces N_+ and N_ϕ (or N_- and N_ϕ) is a triple point in the plane μ, λ if the roots $z = e^{\pm i\phi}$ and $z = 1$ (or $z = -1$) are simple at this point and if the intersecting surfaces do not come in contact either with the parameter plane or with each other.

Here and below the analytic conditions for determining the signs of the parameter changes which lead into the domain with the largest number of roots inside the unit circle will not be given here because of the difficulty of their practical application. In these cases it is more expedient to consider $\chi(z)$ on the N-surfaces or their intersections directly.

6°. A point of the surface N_+ or N_- is a triple point in the parameter space μ, λ, ν if at this point

$$\chi_z' = 0, \qquad \chi_{zz}'' = 0, \qquad \chi_{zzz}''' \neq 0$$

$$\begin{vmatrix} \chi_\mu' & \chi_\lambda' \\ \chi_{z\mu}'' & \chi_{z\lambda}'' \end{vmatrix}^2 + \begin{vmatrix} \chi_\lambda' & \chi_\nu' \\ \chi_{z\lambda}'' & \chi_{z\nu}'' \end{vmatrix}^2 + \begin{vmatrix} \chi_\nu' & \chi_\mu' \\ \chi_{z\nu}'' & \chi_{z\mu}'' \end{vmatrix}^2 \neq 0 \tag{2.9}$$

3. Example 3.1. Let us construct the stability domain D_3 in the space of coefficients μ, λ, ν of the third-degree characteristic polynomial (*)

$$\chi(z, \mu, \lambda, \nu) = z^3 + \mu z^2 + \lambda z + \nu = 0 \tag{3.1}$$

The equations of the surfaces N_+, N_-, and N_ϕ can be written in accordance with (1.9) as

$$1 + \mu + \lambda + \nu = 0, \qquad -1 + \mu - \lambda + \nu = 0 \tag{3.2}$$
$$f = \nu(\nu - \mu) + \lambda - 1 = 0, \qquad |\mu - \nu| < 2$$

Since one of the determinants of (2.9), i.e.

*) A construction of D_3 with the aid of the Hurwitz criterion will be found in [1].

$$\Delta = \begin{vmatrix} \chi_\mu' & \chi_\nu' \\ \chi_{z\mu}'' & \chi_{z\nu}'' \end{vmatrix} = -2z$$

is different from zero, the stability domain D_3 lies in the neighborhood of the triple points of the surfaces N_+ and N_- corresponding to the triple root $z = 1$ and $z = -1$. The coordinates of these points can be found from Eqs. $\chi = 0$ $\chi_z' = 0$ and $\chi_{zz}'' = 0$. These coordinates turn out to be $\mu = -3$, $\lambda = 3$, $\nu = -1$ (the point M_1) for N_+ and $\mu = 3$, $\lambda = 3$, $\nu = 1$ (the point M_2) for N_-.

The change in the parameter $\lambda = 3 + d\lambda$ in the neighborhood of the above points which leads into D_3 can be determined from the variation of the third root along the lines $\chi = 0$ and $\chi_z' = 0$ corresponding to the double root $z = 1$ and $z = -1$. Eqs. of these lines are

$$\mu = -\frac{\lambda + 3}{2}, \qquad \nu = \frac{1 - \lambda}{2} (\Gamma_+), \qquad \mu = \frac{\lambda + 3}{2}, \qquad \nu = \frac{\lambda - 1}{2} (\Gamma_-)$$

Characteristic polynomial (3.1) on these lines can be written as

$$(z - 1)^2 \left(z - \frac{\lambda - 1}{2}\right) = 0, \qquad (z + 1)^2 \left(z + \frac{\lambda - 1}{2}\right) = 0$$

This implies that the domain D_3 is associated with $d\lambda < 0$.

The sign of the change in the parameter μ associated with entry into D_3 can be found from condition (2.8)) by considering the neighborhood of the points Γ_+ and Γ_- for $\lambda = 3 + d\lambda$ and $\nu = \text{const}$, since $\chi_z' = 1 \neq 0$. In the neighborhood of the point Γ_+ we have $\mu = -3 - d\lambda/2$, and condition (2.8),

$$(\Delta z \chi_\nu' \chi_{zz}'') \, d\mu = -2z^2 (6z + 2\mu) \, d\mu < 0$$

is fulfilled for $d\mu > 0$. In the neighborhood of the point Γ_- ($\mu = 3 + d\lambda/2$) it is fulfilled for $d\mu < 0$. Thus, the domain D_3 lies in the neighborhood of the point M_1 for $d\mu > 0$, $d\lambda < 0$, and in the neighborhood of the point M_2 for $d\mu < 0$, $d\lambda < 0$ (Fig. 1*a* and *b*).

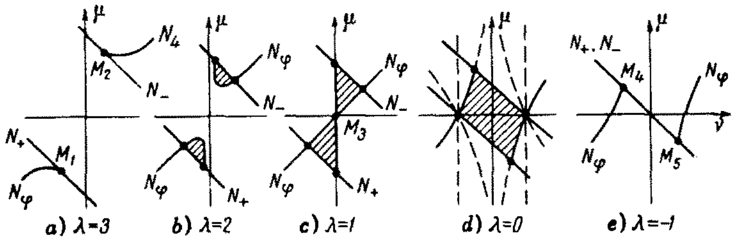

Fig. 1

As the parameter λ decreases, the plane-section domain D_3 becomes simply connected upon the appearance of a double point of the node type on the boundary N_ϕ (3.2) (Fig. 1*c*). The coordinates of the double point M_3 can be found from Eqs. $f = 0$, $f_\mu' = 0$, and $f_\nu' = 0$. They turn out to be $\mu = 0$, $\lambda = 1$, $\nu = 0$. With further decreases in λ the plane section of the stability domain contracts and vanishes at the triple points of intersection of the surfaces N_+, N_-, and N_ϕ: $M_4\{\mu = 1, \lambda = -1, \nu = -1\}$, $M_5\{\mu = -1, \lambda = -1, \nu = 1\}$, i.e. in the cross section $\lambda = -1$ (Fig. 1*d* and *e*).

The segments of N_+, N_-, and N_ϕ which are not boundaries of D_3 appear in Fig. 1 in order better to illustrate the qualitative dependence of the stability domain on the parameters. The broken curves in Fig. 1*d* represent the relationships between μ and ν obtained in constructing the stability domain from the conditions of violation of Schur's conditions. Some of these curves have a singular point belonging to the boundary of D_3.

E x a m p l e 3.2. The following characteristic equation was obtained in an investigation [8] of the stability of motion of an impact damper in symmetrical operation at the resonance frequency with two impacts per period:

$$z^4 + (h - 2g)\, z^3 + (2R - 2h + g^2)\, z^2 + (h - 2Rg)\, z + R^2 = 0$$

$$h = \frac{2\mu\,(1 + R)^2}{(1 + \mu)^2}\left(1 - d + \frac{\pi^2}{8\mu}\right), \qquad g = \frac{(1 + R)(1 - \mu)}{1 + \mu} \tag{3.3}$$

Here R is the factor of velocity restitution after impact $(0 < R < 1)$; μ is the relative mass of the damper $(\mu > 0)$; d is the relative gap between the colliding masses $(d > 0)$.

In accordance with (1.11) we can write the equations of the N-surfaces in the space of the parameters μ, R, and d, i.e.

$$\mu = 0 \tag{3.4}$$

for N_+

$$d = 1 + \frac{\pi^2}{8\mu} - \frac{1}{2\mu} \tag{3.5}$$

for N_-

$$R = 1 \tag{3.6}$$

$$d = 1 + \frac{\pi^2}{8\pi} + \frac{1 + \mu}{2\mu}\left(\frac{1 - R}{1 + R}\right)^2, \qquad \left|\frac{1 - \mu}{1 + \mu}\right| < 1 \tag{3.7}$$

for N_ϕ

In order to determine the position of the stability domain D_4 in the parameter space we consider the neighborhood of intersection of the double surface N_ϕ (3.6) and (3.7),

$$R = 1, \qquad d = 1 + \pi^2 / 8\mu \tag{3.8}$$

and investigate the behavior of (3.3) along some parametric trajectory $\mu =$ const, $d =$ const passing through line (3.8). Since the qualitative picture of disposition of the domains D_k in the neighborhood of (3.8) does not depend on μ in the interval $0 < \mu < \infty$, let us take for simplicity $\mu = 1$, $d = 1 + {}^{\pi}/_{4}$. Eq. (3.3) can then be written as

$$(z^2 + R)^2 = 0$$

Hence, the domain D_4 contains the segment

$$\mu = 1, \quad d = 1 + {}^{1}/_{8}\pi^2, \quad R < 1$$

of the chosen parametric trajectory, and the entire stability domain is isolated in the parameter space by the inequalities

$$0 < R < 1, \quad \mu > 0, \quad -\frac{1}{2\mu} < d - 1 - \frac{\pi^2}{8\mu} < \frac{1 + \mu}{2\mu}\left(\frac{1 - R}{1 + R}\right)^2$$

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